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LETTER TO THE EDITOR

Path integral bosonisation for the Thirring model in the presence of vortices

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Abstract. We study the bosonisation of the Abelian Thirring model in the presence of vortices in the framework of path integrals.

Recently the bosonisation of two-dimensional fermionic models in the framework of path integrals has been shown to be a powerful non-perturbative technique for exactly solving these models in the absence of fermion zero modes [1, 2].

It is the purpose of this letter to implement the path integral bosonisation framework for the case of non-trivial fermion zero modes by bosonisation of the Abelian Thirring model in the presence of vortex field configurations.

Let us start our analysis by considering the Abelian Thirring model interacting with a vortex field with topological charge n. Its Euclidean Lagrangian is given by

$$\mathscr{L}_0(\psi, \bar{\psi}, A_\mu^{(n)}) = \bar{\psi}\, \mathrm{i}\gamma_\mu \partial_\mu \psi + \tfrac{1}{2} g^2 (\bar{\psi}\gamma_\mu\psi)^2 + e\bar{\psi}\gamma_\mu A_\mu^{(n)}\psi \tag{1}$$

where (g, e) are positive model coupling constants. The Euclidean Hermitian γ_μ matrices we are using satisfy the relations

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \qquad \gamma_\mu\gamma_5 = \mathrm{i}\varepsilon_{\mu\nu}\gamma_\nu \qquad \gamma_5 = \mathrm{i}\gamma_0\gamma_1 \qquad \varepsilon_{01} = -\varepsilon_{10} = 1 \tag{2}$$

and the vortex field $A_\mu^{(n)}(x)$ with topological charge n (Chern number) and 'length' R is

$$A_\mu^{(n)}(x) = n\frac{1}{R^2 + x^2}\varepsilon_{\lambda\mu}x_\lambda . \tag{3}$$

The generating functional associated with the theory (1) is, thus, given by

$$Z[\eta, \bar{\eta}, A_\mu^{(n)}] = \int D[\psi]\, D[\bar{\psi}] \exp\left(-\int \mathrm{d}^2x (\mathscr{L}_0(\psi, \bar{\psi}, A_\mu^{(n)}) + \bar{\eta}\psi + \bar{\psi}\eta)(x)\right). \tag{4}$$

In order to implement the path integral bosonisation gauge invariant technique [1, 2], we rewrite the fermion interaction term in the Hubbard–Stratonovitch form:

$$\begin{aligned} Z[\eta, \bar{\eta}, A_\mu^{(n)}] = \int & \mathrm{D}[\psi]\, \mathrm{D}[\bar{\psi}]\, \mathrm{D}[B_\mu] \exp\left(-\tfrac{1}{2}\int \mathrm{d}^2x\, B_\mu^2(x)\right) \\ & \times \exp\left(-\int \mathrm{d}^2x [\bar{\psi}\, \mathrm{i}\gamma_\mu(\partial_\mu + B_\mu - \mathrm{i}eA_\mu^{(n)})\psi](x)\right) \\ & \times \exp\left(-\int \mathrm{d}^2x (\bar{\psi}\eta + \bar{\eta}\psi)(x)\right). \end{aligned} \tag{5}$$

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Let us now proceed as in [1, 2] by making the (partial) decoupling field change in (5)

$$\psi(x) = [\exp(\mathrm{i}g\gamma_5 u)\chi](x) \qquad \bar{\psi}(x) = [\chi \exp(\mathrm{i}g\gamma_5 u)](x)$$
$$B_\mu(x) = (g\varepsilon_{\mu\nu}\partial_\nu u)(x). \tag{6}$$

As has been shown by Fujikawa [3], the transformations of (6) are not free of cost in the fermionic sector since the functional measure $\mathrm{D}[\psi]\,\mathrm{D}[\bar{\psi}]$ is defined in terms of the normalised eigenvectors of the Dirac operator $\not{\mathrm{D}}(B_\mu; A_\mu^{(n)})$ in the presence of the external (auxiliary) Abelian field B_μ and of the vortex topological field $A_\mu^{(n)}$.

At this point we note that after the chiral change takes place, the new quantum fermion vacuum is defined by the fermionic theory in the presence of the topological vortex; i.e. $\mathrm{D}[\chi]\,\mathrm{D}[\bar{\chi}]$ is now defined in terms of the eigenvectors of the Dirac operator $\not{\mathrm{D}}(A_\mu^{(n)})$ which in turn has precisely n fermionic zero modes with definite chiriality [4]. Their explicit expressions are

$$\psi_{(0),l}(x) = \left(\frac{1}{2\pi R}\right)^{1/2}\left(\frac{x_1 + \mathrm{i}x_2}{R}\right)^{l-1}\begin{pmatrix} h_-(x, A_\mu^{(n)}) \\ 0 \end{pmatrix} \tag{7a}$$

$$\gamma_5\psi_{(0),l}(x) = \psi_{(0),l}(x) \tag{7b}$$

$$h_\pm(x, A) = \exp\left(\mathrm{i}e \int \mathrm{d}^2z\ \Delta_{m=0}(x - z)(\partial_\mu A_\mu^{(n)} \pm \mathrm{i}\varepsilon_{\mu\nu}\partial_\mu A_\nu^{(n)})(z)\right) \tag{7c}$$

with $\Delta_{m=0}(z)$ the (infrared regularised) massless Klein-Gordon propagator.

The associate Jacobians are given by [1, 2]

$$\mathrm{D}[\psi]\,\mathrm{D}[\bar{\psi}] = \mathrm{D}[\chi]\,\mathrm{D}[\bar{\chi}]\frac{\det \not{\mathrm{D}}(B_\mu, A_\mu^{(n)})}{\det \not{\mathrm{D}}(B_\mu \equiv 0, A_\mu^{(n)})} \tag{8a}$$

$$\mathrm{D}[B_\mu] = \mathrm{D}[u]. \tag{8b}$$

So, we face the problem of the evaluation of the ratio of the two Dirac determinants with zero modes.

By following the procedure of [1, 2], we first introduce a one-parameter continuous family of Dirac operators interpolating the pure vortex Dirac operator and $\not{\mathrm{D}}(A_\mu^{(n)}, B_\mu) = \mathrm{i}\gamma_\mu(\partial_\mu + gB_\mu + \mathrm{i}eA_\mu^{(n)})$ defined by the expression

$$\not{\mathrm{D}}^{(\sigma)}(B_\mu, A_\mu^{(n)}) = \exp(\mathrm{i}g\gamma_5\sigma u)\not{\mathrm{D}}(A_\mu^{(n)})\exp(\mathrm{i}g\gamma_5\sigma u) \qquad (0 \leqslant \sigma \leqslant 1). \tag{9}$$

By using a proper-time prescription to define the functional determinant of $\not{\mathrm{D}}^{(\sigma)}(B_\mu, A_\mu^{(n)})$ (after making the analytic extension $g = -\mathrm{i}\bar{g}$ [2]). We have the result

$$\log\det\not{\mathrm{D}}^{(\sigma)} = \tfrac{1}{2}\log\det(\not{\mathrm{D}}^{(\sigma)^2})|_{\bar{g}=\mathrm{i}g}$$
$$= -\lim_{\varepsilon\to 0^+}\int_\varepsilon^{1/\varepsilon}\frac{\mathrm{d}\zeta}{\zeta}\mathrm{Tr}[\exp(-\zeta\not{\mathrm{D}}^{(\sigma)^2})(\mathbb{1} - \mathbb{P}^{(\sigma)})] \tag{10}$$

with

$$\mathbb{P}^{(\sigma)}|\zeta\rangle = \sum_{l=1}^{n}\langle\zeta|\beta_{(0),l}^{\sigma}\rangle\beta_{(0),l}^{\sigma}$$

denoting the projection over the zero modes of the Dirac operator $\not{\mathrm{D}}^{(\sigma)}(B_\mu, A_\mu^{(n)})$. Let us remark that by the Atyah-Singer theorem, this operator still has n zero modes

$\beta^{\sigma}_{(0),l}(x)$ which are related to those of (7a) by an analytically continued chiral rotation (6):

$$\beta^{\sigma}_{(0),l}(x) = \exp(-\bar{g}\gamma_5\sigma u)\psi_{(0),l}(x). \tag{11}$$

The functional determinant (10) satisfies the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\log\det \mathbb{D}^{(\sigma)}(B_\mu, A_\mu^{(n)})$$
$$= -2\lim_{\varepsilon\to 0^+}\mathrm{Tr}[\bar{g}\gamma_5 u \exp(-\zeta\,\mathbb{D}^{(\sigma)^2})(\mathbb{1}-\mathbb{P}^{(\sigma)})]|_{\varepsilon}^{1/\varepsilon}$$
$$+\lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{1/\varepsilon}\frac{\mathrm{d}\zeta}{\zeta}\mathrm{Tr}\left[\exp(-\zeta\,\mathbb{D}^{(\sigma)^2})\frac{\mathrm{d}}{\mathrm{d}\sigma}\mathbb{P}^{(\sigma)}\right]. \tag{12}$$

The second term in the right-hand side of (12) is easily evaluated by noting that

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\mathbb{P}^{(\sigma)}|\zeta\rangle = \sum_{l=1}^{m}\langle\zeta|-\bar{g}\gamma_5\phi\beta^{\sigma}_{(0),l}\rangle\beta^{\sigma}_{(0),l} + \sum_{l=1}^{n}\langle\zeta|\beta^{\sigma}_{(0),l}\rangle - \bar{g}\gamma_5\phi\beta^{\sigma}_{(0),l} \tag{13}$$

which yields the zero-mode contribution for the determinant (12):

$$\lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{1/\varepsilon}\frac{\mathrm{d}\zeta}{\zeta}\mathrm{Tr}\left(\exp(-\zeta\,\mathbb{D}^{(\sigma)^2})\frac{\mathrm{d}}{\mathrm{d}\sigma}\mathbb{P}^{(\sigma)}\right) = \lim_{\varepsilon\to 0^+}\left[4\log\left(\varepsilon\bar{g}\sum_{l=1}^{n}\langle\beta^{\sigma}_{(0),l}|\phi|\beta^{\sigma}_{(0),l}\rangle\right)\right]. \tag{14}$$

Since $\mathbb{D}^{(\sigma)^2}$ is a self-adjoint invertible operator in the manifold orthogonal to its zero modes, we can use the usual Seeley–De Witt technique to evaluate the first term in (12), which produces the usual result

$$-\mathrm{Tr}(\bar{g}\gamma_5 u\exp(-\zeta\,\mathbb{D}^{(\sigma)^2})(\mathbb{1}-\mathbb{P}^{(\sigma)}))|_{\varepsilon}^{1/\varepsilon}$$
$$=\sigma\frac{1}{\pi}\mathrm{Tr}\left((-\mathrm{i}\bar{g})u(-\mathrm{i}\bar{g})\partial^2 u + \frac{\varepsilon_{\mu\nu}}{2}F_{\mu\nu}(A_\mu^{(n)})\right). \tag{15}$$

By combining (15) with (14) and coming back to the real coupling constant g, we obtain the final expression for the Jacobian (8a) after integrating the differential equation (12)

$$\log\det\mathbb{D}(B_\mu, A_\mu^{(n)}) - \log\det\mathbb{D}(B_\mu = 0, A_\mu^{(n)})$$
$$=\frac{g^2}{\pi}\int \mathrm{d}^2x\,\tfrac{1}{2}(\partial u)^2 + \frac{g}{\pi}\int \mathrm{d}^2x\,\varepsilon_{\mu\nu}F_{\mu\nu}(A_\mu^{(n)})u$$
$$-\mathrm{i}gC(\varepsilon)\sum_{l=1}^{n}\int \mathrm{d}^2x\,\psi^{\dagger}_{(0),l}(x)u(x)\psi_{(0),l}(x) \tag{16}$$

where we have used the unitarity of the matrix $\exp\{\mathrm{i}g\gamma_5 u\}$ to evaluate (14) and $C(\varepsilon)$ is the usual infrared divergence contribution for the zero-mode term, which can be made finite by a multiplicative renormalisation of the Thirring coupling constant g.

The generating functional thus takes the simple form

$$Z[\eta,\bar{\eta},A_\mu^{(n)}]$$
$$=\int \mathrm{D}[u]\exp\left\{-\frac{1}{2}\int \mathrm{d}^2x\left[\left(1-\frac{g^2}{\pi}\right)(\partial u)^2\right](x)\right.$$
$$-\frac{g}{\pi}\int \mathrm{d}^2x(\varepsilon_{\mu\nu}F_{\mu\nu}(A_\mu^{(n)})u)(x)$$
$$\left.\times\exp\left(-\mathrm{i}g\int \mathrm{d}^2x\,\psi^{\dagger}_{(0),l}(x)u(x)\psi_{(0),l}(x)\right)\right\}Z_{(0)}[\eta,\bar{\eta},A_\mu^{(n)},u] \tag{17}$$

with $Z_{(0)}[\eta, \bar{\eta}, A_\mu^{(n)}, u]$ being the generating functional for the fermions in the pure vortex field configuration

$$Z_{(0)}[\eta, \bar{\eta}, A_\mu^{(n)}, u] = \int D[\chi]\, D[\bar{\chi}]$$

$$\times \exp\left(-\int d^2x(\bar{\chi}\, \not{D}(A_\mu^{(n)})\chi + \bar{\chi} \exp ig\gamma_5 u\eta + \bar{\eta} \exp ig\gamma_5 u\chi)(x)\right). \tag{18}$$

Let us exemplify our approach by calculating the two-point fermion correlation function for an external vortex field with topological charge 1. By functional differentiation of $Z_{(0)}[\eta, \bar{\eta}, A_\mu^{(n)}, u]$ twice, we get

$$(\bar{\chi} \exp ig\gamma_5 u)_\alpha(x)(\exp ig\gamma_5 u\chi)_\beta(y)$$

$$= [\exp(ig\gamma_5 u(x))S^{(1)}(x-y) \exp(ig\gamma_5 u(y))]_{\alpha\beta} \tag{19}$$

where $S^{(1)}(x-y)$ is the Euclidean Green function of the Dirac operator $\not{D}(A_\mu^{(n)})$ for $n=1$ which is given explicitly by [4]

$$S^{(1)}(x-y) = -\frac{1}{2\pi}\begin{bmatrix} 0 & \left(\frac{1+y^2}{1+x^2}\right)^{1/2}\frac{x_1 - ix_2 - (y_1 - iy_2)}{(x-y)^2} + \frac{(y_1 - iy_2)}{(1+x^2)^{1/2}(1+y^2)^{1/2}} \\ -\left(\frac{1+x^2}{1+y^2}\right)^{1/2}\frac{x_1 + ix_2 - (y_1 + iy_2)}{(x-y)^2} + \frac{x_1 + ix_2}{(1+x^2)^{1/2}(1+y^2)^{1/2}} & 0 \end{bmatrix}. \tag{20}$$

By evaluation of the u average of (19) we finally have the complete expression for $\langle\bar{\psi}(x)\psi|y)\rangle$:

$$\langle\bar{\psi}(x)\psi(y)\rangle = \exp\left(\frac{1}{2}\frac{1}{1-g^2/\pi}\int d^2z\, d^2\bar{z}\, J_\mu(z; [x,y])\right.$$

$$\left.\times \Delta_{m=0}(z-\bar{z})J_\mu(\bar{z}; [x,y])S^{(1)}(x-y)\right) \tag{21}$$

where

$$J_\mu(z; [x,y]) = \left(\frac{y}{\pi}\varepsilon_{\mu\nu}F_{\mu\nu}(A_\mu^{(1)}(z)) - ig\psi^\dagger_{(0),0}(z)\psi_{(0),0}(z)\right.$$

$$\left. + (ig\gamma_5^{(x)}\delta(z-x) + ig\gamma_5^{(y)}\delta(z-y))\right) \tag{22}$$

with

$$\psi_{(0),0}(z) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{\sqrt{1+z^2}}, 0\right)$$

being the only zero mode of $\not{D}(A_\mu^{(1)})$.

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